Dedekind’s Theory of Real Numbers

Intro

Let $\mathbf{Q}$ be the set of rational numbers. It is well known that $\mathbf{Q}$ is an ordered field and also the set $\mathbf{Q}$ is equipped with a relation called “less than” which is an order relation. Between two rational numbers there exists an infinite number of elements of $\mathbf{Q}$. Thus, the system of rational numbers seems to be dense and so apparently complete. But it is quite easy to show that there exist some numbers (?) (e.g., $\sqrt{2}, \sqrt{3} \dots$ etc.) which are not rational. For example, let we have to prove that $\sqrt{2}$ is not a rational number or in other words, there exist no rational number whose square is 2. To do that if possible, purpose that $\sqrt{2}$ is a rational number. Then according to the definition of rational numbers $\sqrt{2} = \frac{p}{q}$, where p & q are relatively prime integers. Hence, $\left(\sqrt{2}\right)^2 = p^2/q^2$ or $p^2 = 2q^2$. This implies that p is even. Let $p = 2m$, then $(2m)^2 = 2q^2$ or $q^2 = 2m^2$. Thus q is also even if 2 is rational. But since both are even, they are not relatively prime, which is a contradiction. Hence $\sqrt{2}$ is not a rational number and the proof is complete. Similarly we can prove that why other irrational numbers are not rational. From this proof, it is clear that the set $\mathbf{Q}$ is not complete and dense and that there are some gaps between the rational numbers in form of irrational numbers. This remark shows the necessity of forming a more comprehensive system of numbers other that the system of rational number. The elements of this extended set will be called a real number. The following three approaches have been made for defining a real number.

1. Dedekind's Theory
2. Cantor's Theory
3. Method of Decimal Representation

The method known as Dedekind's Theory will be discussed in this not, which is due to R. Dedekind (1831-1916). To discuss this theory we need to work on the following definitions:

Rational number

A number which can be represented as $\frac{p}{q}$ where p is an integer and q is a non-zero integer i.e., $p \in \mathbf{Z}$ and $q \in \mathbf{Z} \setminus \{0\}$ and p and q are relatively prime as their greatest common divisor is 1, i.e., $(p, q) = 1$.

Ordered Field

Here, $\mathbf{Q}$ is, an algebraic structure on which the operations of addition, subtraction, multiplication & division by a non-zero number can be carried out.

Least or Smallest Element

Let $A \subseteq Q$ and $a \in Q$. Then a is said to be a least element of A if (i) $a \in A$ and (ii) $a \leq x$ for every $x \in A$.

Greatest or Largest Element

Let $A \subseteq Q$ and $b \in Q$. Then b is said to be a least element of A if (i) $b \in A$ and (ii) $x \leq b$ for every $x \in A$.

Dedekind's Section (Cut) of the Set of All the Rational Numbers

Since the set of rational numbers is an ordered field, we may consider the rational numbers to be arranged in order on straight line from left to right. Now if we cut this line by some point P , then the set of rational numbers is divided into two classes L and U . The rational numbers on the left, i.e. the rational numbers less than the number corresponding to the point of cut P are all in L and the rational numbers on the right, i.e. The rational number greater than the point are all in

U . If the point P is not a rational number then every rational number either belongs to L or U . But if P is a rational number, then it may be considered as an element of U .

Definitions derived:

Real Numbers

Let $L \subset \mathbf{Q}$ satisfying the following conditions:

1. L is non-empty proper subset of $\mathbf{Q}$.
2. $a, b \in \mathbf{Q}$, $a < b$ and $b \in L$ then this implies that $a \in L$.
3. L doesn't have a greatest element.

Let $U = \mathbf{Q} - L$. Then the ordered pair $< L, U >$ is called a section or a cut of the set of rational numbers. This section of the set of rational numbers is called a real number.

Notation: The set of real numbers $\alpha, \beta, \gamma, \ldots$ is denoted by $\mathbf{R}$.

Let $\alpha = \langle L, U \rangle$ then L and U are called Lower and Upper Class of α respectively. These classes will be denoted by $L(\alpha)$ and $U(\alpha)$ respectively.

Remark

From the definition of a section $\langle L, U\rangle$ of rational numbers, it is clear that $L \cup U = \mathbf{Q}$ and $L \cap U = \phi$. Thus a real number is uniquely determined iff its lower class L is known.

For Example

Let $L = \{x : x \in \mathbf{Q}, x \leq 0 \; or \; x > 0, \text{ and } x^2 < 2\}$ Then prove that L is a lower class of a real number.

Proof: $\star$ Since $0 \in L$ and $2 \notin L \Rightarrow L$ is non-empty proper subset of $\mathbf{Q}$.
$\star\star$ Let $a, b \in \mathbf{Q}, a > b$ and $b \in L$. If $a \leq 0$ then $a \in L$. If $a > b >$ so $b \in \Rightarrow b^2 < 2 \Rightarrow a^2 < b^2 < 2 \Rightarrow a \in L$.

$\star\star\star$ Let $a \in \mathbf{Q}$ s.t. $a \in L$. If $a \leq 0$ then $a \in L$. If $a > 0$ then $a^2 < 2$.
Let for $b > 0$ and $b \in \mathbf{Q}$,$b = \dfrac{m+na}{o+pa}$ for any $m \geq n \geq o \geq p \in \mathbf{Z}$.
Also, $b - a = \dfrac{m+na}{o+pa} - a$

$$= \frac{m + na - oa - pa^2}{o+pa}$$

$$= \frac{m + (n-o)a - pa^2}{o+pa} > 0$$

$\Rightarrow b > a$.
And similarly, $2 - b^2 > 0, \Rightarrow b^2 < 2$.
Thus, $0 < a < b$ and $b^2 < 2 \Rightarrow b \in L$. Hence L has no greatest element.
Since, L satisfies all the conditions of a section of rational numbers, it is a lower class of a real number. [Proved]
Remark: In the given problem, U is an upper class of a real number given by the set
$U = \{x : x \in \mathbf{Q}, x > 0 \text{ and } x^2 > 0\}$, since it has no smallest element.

Real Rational Number

The real number $\alpha = \langle L, U \rangle$ is said to be a real rational number if its upper class U has a smallest element. If r is the smallest element of U , then we write $\alpha = r^*$.

Irrational Number

The real number $\alpha = \langle L, U \rangle$ is said to be an irrational number if U does not have a smallest element.

Important Results

If $\langle L, U \rangle$ is a section of rational numbers, then

1. U is a non-empty proper subset of $\mathbf{Q}$.
2. $a, b \in \mathbf{Q},\ a < b\ and\ a \in U \Rightarrow b \in U$.
3. $a \in L, b \in U \Rightarrow a < b$.
4. if k is a positive rational number, then there exists $x \in L\ y \in U$ such that $y - x = k$
5. if L contains some positive rational numbers and $k > 1$ then there exists $x \in L$ and $y \in U$ such that $\frac{y}{x} = k$.